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EIGENSENSITIVITY AND STRUCTURAL OPTIMIZATION WITH ACCENT ON THE REPEATED FREQUENCIES

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Abstract

Sensitivity analysis is the study of changes in system response with respect to design parameters. It is being used in a variety of engineering disciplines ranging from automatic control theory to the analysis of large-scale physiological systems. Some of areas where sensitivity analysis has been applied include: development of insensitive control systems, use in gradient-based mathematical programming methods and approximation of system response to change in a system parameter, assessment of design changes on system performance. Eigensensitivity analysis represents a collection of mathematical methods for analyzing structures which is, within dynamical modification, related to sensitivity of eigenvalues and eigenvectors. Therefore, the application of sensitivity analysis is limited to construction of segments for which necessary mathematical relations can be determined. If this is not possible, sensitivity analysis is only partially applicable. Eigenvalue sensitivity analysis is useful when resonant frequencies need to be restricted. Different methods for analyzing structural eigensensitivity are considered. An example is given to illustrate an optimization problem for a system with repeated frequencies.

Key words: eigensensitivity, structural optimization, repeated frequencies

1. INTRODUCTION

Dynamical analysis of structures can easily be conducted via finite elements modeling. Therefore, while finite element analysis method is highly adequate for modeling complex structures, one of its major drawbacks lies in the usage of a large number of degrees of freedom in calculating the exact eigenpairs. This number can amount to few tens of thousands, or even more. To reduce the calculation time it is possible to divide the complex structure into connected substructures and analyze each one separately. The dynamical behavior of each substructure is represented only by a reduced set of eigenpairs of interest, which contributes to significant problem simplification.

2. FIRST ORDER DESIGN SENSITIVITY OF EIGENVECTORS CORRESPONDING TO SIMPLE EIGENVALUES

In order to obtain the simplest possible derivation of eigenvalue design sensitivity in this setting, a basis $\phi^i\left(i=1,...,m\right)$ of Z may be introduced. It presumed that kinematic constraints do not depend explicitly on design, so the vectors ϕ^i are independent of design. Recall that the dimension of the space $Z \subset R^n$ of kinematically admissible displacements is $m < n$. Any vector $y_g \in Z$ may be written as linear combination of the ϕ^i, that is,

$$y_g = \sum_{i=1}^{m} q_i \phi^i = Q_q$$

where $\{Q\} = \left\{\phi^1 \ \phi^2 ... \ \phi^m\right\}$ and the coefficients q_i are to be determined.
In matrix form, the eigenvalue problem of a mechanical system can be expressed as:

$$\left[K(b)\right]\{q\}_r = \zeta_r \left[M(b)\right]\{q\}_r \tag{1}$$

where the eigenvector $\{q\}$ is normalized by the condition

$$\{q\}_r^T \left[M(b)\right]\{q\}_r = 1 \tag{2}$$

It is presumed that the reduced global stiffness and mass matrices are positive definite and differenciable with respect to design parameters and could be written:

$$\left[K(b)\right] = \{Q\}^T \left[K_g(b)\right]\{Q\}_r \quad \text{and} \quad \left[M(b)\right] = \{Q\}^T \left[M_g(b)\right]\{Q\}_r$$

The matrix $\{Q\} = \left\{\phi^1 \ \phi^2 ... \ \phi^m\right\}$ does not depend on design parameters, than the matrices $\left[K(b)\right]$ and $\left[M(b)\right]$ are differentiable with respect to design if $\left[K_g(b)\right]$ and $\left[M_g(b)\right]$ are
Differentiating Eq. (1) with respect to an updating variable b_i gives

$$\left(\frac{\partial\left[K(b)\right]}{\partial b_i} - \frac{\partial \zeta_r}{\partial b_i}\left[M(b)\right] - \zeta_r \frac{\partial\left[M(b)\right]}{\partial b_i}\right)\{q\}_r + \left(\left[K(b)\right] - \zeta_r\left[M(b)\right]\right)\frac{\partial\{q\}_r}{\partial b_i} = \{0\} \tag{3}$$

Pre-multiplying (3) by $\{\phi\}_r^T$ leads to

$$\frac{\partial \zeta_r}{\partial b_i} = \{q\}_r^T \frac{\partial[K]}{\partial b_i}\{q\}_r - \zeta_r \{q\}_r^T \frac{\partial[M]}{\partial b_i}\{q\}_r \tag{4}$$

The equation (4) includes only the eigenvalue and eigenvector under consideration, therefore a complete solution of eigenproblem is not needed to obtain these derivatives. The eigenvector derivatives can be expressed as linear combinations of all eigenvectors of the system if the eigenvalues are assumed distinct, because *N* eigenvectors are linearly independent and they can be used as a set of basis vectors for spanning *N* - dimensional space. Thus,

$$\frac{\partial\{q\}_r}{\partial b_i} = \sum_{j=1}^{N} c_{rj}^i \{q\}_j \, . \tag{5}$$

Substituting (4) into (2) and pre-multiplying Eq. (2) by $\{q\}_k^T$

$$\{q\}_k^T\left(\frac{\partial[K]}{\partial b_i} - \frac{\partial \zeta_r}{\partial b_i}[M] - \zeta_r \frac{\partial[M]}{\partial b_i}\right)\{q\}_r + \{q\}_k^T\left([K] - \zeta_r[M]\right)\sum_{j=1}^{N} c_{rj}^i \{q\}_j = \{0\} \tag{6}$$

If $k \neq r$,

$$\{q\}_k^T\left(\frac{\partial[K]}{\partial b_i} - \zeta_r \frac{\partial[M]}{\partial b_i}\right)\{q\}_r + c_{rk}^i\left(\zeta_k - \zeta_r\right) = 0 \tag{7}$$

Thus

$$c_{rk}^i = \frac{\{q\}_k^T\left(\frac{\partial[K]}{\partial b_i} - \lambda_r \frac{\partial[M]}{\partial b_i}\right)\{q\}_r}{\zeta_r - \zeta_k} \tag{8}$$

c_{rr}^{i} can be obtained by differentiating (2) with respect to parameter b_i

$$2\{q\}_r^T[M]\frac{\partial\{q\}_r}{\partial b_i}+\{q\}_r^T\frac{\partial[M]}{\partial b_i}\{q\}_r=0 \tag{9}$$

Substituting (5) into (9) leads to

$$c_{rk}^{i}=-\frac{1}{2}\{q\}_r^T\frac{\partial[M]}{\partial b_i}\{q\}_r, \qquad \frac{\partial\{q\}_r}{\partial b_i}=\sum_{j=1}^{N}c_{rj}^{i}\{q\}_j \tag{10}$$

Using the Rayleigh quotient representation of eigenvalues, it is well known [1] that the second eigenvalue of the problem minimizes the Rayleigh quotient over all vectors $\{q\}\in W$, where is W subspace of Z. Rayleigh's method for computing the eigenvalues of a conservative system utilizes an assumed mode for sinusoidal motion and then equates the maximum kinetic energy to the maximum potential energy. That is

$$E_{k,\max}=E_{p,\max} \text{ or } \frac{1}{2}\{q\}_r^T[M]\{q\}_r=\zeta\frac{1}{2}\{q\}_r^T[K]\{q\}_r \tag{11}$$

Since the second eigenvalue is strictly larger than the smallest simple eigenvalue ζ,

$$\zeta<\frac{\{q\}_r^T[K]\{q\}_r}{\{q\}_r^T[M]\{q\}_r} \text{ for all } \{q\}\in W,\ \{q\}\neq\{0\} \tag{12}$$

or

$$\zeta<\frac{\{q\}_r^T[K]\{q\}_r}{\{q\}_r^T[M]\{q\}_r} \text{ for all } \{q\}\in W,\ \{q\}\neq\{0\} \tag{13}$$

Several numerical techniques exist for solving Eq (4). Nelson [2] presented a direct computational technique that uses the reduced global stiffness matrix and is effective for computations in which the reduced system matrices are known. Potential exists for direct application of numerical techniques such as subspace iteration to construct a solution of Eq (6), in conjunction with solution of the basic eigenvalue problem.

3. SECOND ORDER DESIGN SENSITIVITY OF EIGENVECTORS CORRESPONDING TO SIMPLE EIGENVALUES

The ith component of the gradient of the smallest eigenvalue ζ with respect to design may be written from Eq.(4). Differentiating with respect to b_i gives

$$\begin{aligned}\frac{\partial^2\zeta_r}{\partial b_i\partial b_j}&=\frac{\partial^2}{\partial b_i\partial b_j}\Big[\{q\}_r^T[K]\{q\}_r\Big]-\zeta_r\frac{\partial^2}{\partial b_i\partial b_j}\Big[\{q\}_r^T[M]\{q\}_r\Big]-\\&-\left\{\frac{\partial}{\partial b_j}\Big[\{q\}_r^T[K]\{q\}_r\Big]-\zeta_r\frac{\partial}{\partial b_j}\Big[\{q\}_r^T[M]\{q\}_r\Big]\right\}\frac{\partial}{\partial b_i}\Big[\{q\}_r^T[M]\{q\}_r\Big]+\\&+2\frac{\partial}{\partial b_i}\Big[\{q\}_r^T[M]\{q\}_r\Big]\frac{\partial\{q\}_r}{\partial b_j}-2\zeta_r\frac{\partial}{\partial b_i}\Big[\{q\}_r^T[M]\{q\}_r\Big]\frac{\partial\{q\}_r}{\partial b_j}\end{aligned} \tag{14}$$

In order to evaluate the second derivative of ζ in Eq. (14) $\partial\{q\}_r/\partial b_i$ and $\partial\{q\}_r/\partial b_j$ must be calculated. This may be done solving Eq. (4). Once Eq (4) is solved, the result may be substituted into Eq. (14) to obtain the second design derivative of ζ with respect to design parameters b_i and b_j. The computation of all second design derivatives of ζ requires solution of Eq (4) for $j = 1,...,k$. These results may be substituted into Eq. (14) and the partial derivatives with respect to b_i $(i = 1,...,k)$ may be calculated. Thus, all $k^2/2 + k/2$ distinct derivatives of ζ are obtained with respect to design. In doing so, k sets of equations in Eq. (4) must be dealt with and numerical computation performed to evaluate the right side of Eq. (14). While this is a substantial amount of computation, availability of second design derivative of eigenvalues with respect to design can be of value in iterative design optimization.

4. SYSTEMATIC OCCURRENCE OF REPEATED EIGENVALUES IN STRUCTURAL OPTIMIZATION

In carrying out vibration and buckling analysis of structures, it is well known that computational difficulties can arise if repeated eigenvalues arise. The situation of repeated frequencies, or identical frequencies with different mode shapes, occurs in many physical systems. While repeated eigenvalues may indeed be unlikely in randomly specified structures, they become far more likely in optimized structures.

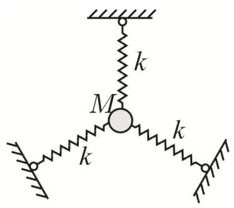

Fig. 1 Two-dimensional vibrations of symmetrically supported mass

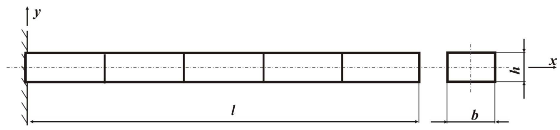

Fig. 2 Vibration of a cantilevered beam

Thompson and Hunt [3] have devoted considerable attention to designs that are constructed with simultaneous buckling failure modes (i.e., repeated eigenvalues). Olhoff and Rasmussen [4] showed that a repeated buckling load may occur in an optimized clamped-clamped column. Ojalvo [5] has given an efficient computation procedure, which preserves the bandwith for very large systems.

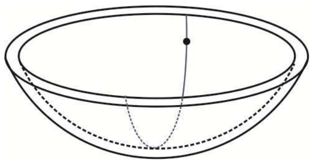

Fig. 3 Vibrations of the frictionless mass in the shallow elliptic dish

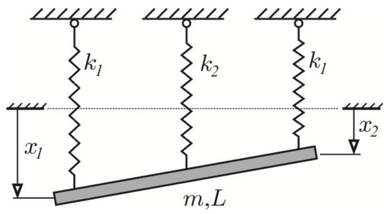

Fig. 4 Spring-mass system with two degrees of freedom

The most common circumstances under which multiple eigenvalues occur in engineering are instances where system symmetry exists, such as structures with two or more planes of reflective or cyclic symmetry or in the limiting case of axisymmetric bodies. Examples of structures with repeated roots are shown in Fig 1-2. Crandall [6] has presented a simple example to explain this phenomenon of repeated frequencies, in physical terms, through cosideration if a frictionless particle sliding back and forth nears the bottom of a shallow elliptic bowl (Fig. 3).

5. VIBRATION EXAMPLE

Consider the spring-mass system shown in Fig. 4 where is the connecting bar of length L, mass M. Selecting the generalized coordinates x_1 and x_2, the system kinetic and potential energies E_k and E_p, respectively, are for small displacements, as

$$E_k = \frac{1}{2}M\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2 + \frac{1}{2}\frac{1}{12}ML^2\left(\frac{\dot{x}_1-\dot{x}_2}{L}\right)^2$$
$$E_p = \frac{1}{2}k_1\left(x_1+f_{st,1}\right)^2 + \frac{1}{2}k_2\left(\frac{x_1+x_2}{2}+f_{st,2}\right)^2 + \frac{1}{2}k_1\left(x_2+f_{st,3}\right)^2 + Mg\frac{x_1+x_2}{2} + const \tag{15}$$

Applying Lagrange's equations of the second order

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{x}_1} + \frac{\partial E_p}{\partial x_1} = 0 \quad \text{and} \quad \frac{d}{dt}\frac{\partial E_k}{\partial \dot{x}_2} + \frac{\partial E_p}{\partial x_2} = 0, \tag{16}$$

it is possible to get differential equations of the motions in matrix form:

$$\begin{bmatrix} \frac{M}{3} & \frac{M}{6} \\ \frac{M}{6} & \frac{M}{3} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_1 \end{Bmatrix} + \begin{bmatrix} k_1+\frac{k_2}{4} & \frac{k_2}{4} \\ \frac{k_2}{4} & k_1+\frac{k_2}{4} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{17}$$

and using Eq (1), the eigenvalue equation for this case has form

$$\begin{bmatrix} 4k_1+k_2 & k_2 \\ k_2 & 4k_1+k_2 \end{bmatrix}\{q\} = \zeta\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\{q\} \tag{18}$$

where $\{q\}$ is amplitude vector and $\zeta = \frac{2M\omega^2}{3}$. Moment inertia of the bar for longitudinal axis is $I_z = \frac{ML^2}{12}$ and ω is angular frequency. Horizontal motion of the bar is ignored and the spring constants (stiffness) are regarding as design variables, i.e. $b_1 = k_1$ and $b_2 = k_2$. The optimal design objective is to find design parameters k_1 and k_2 to minimize weight of the spring supports, which is presumed to be of the form

$$\psi_o = c_1 b_1 + c_2 b_2 \tag{19}$$

where c_1 and c_2 are known constants, while ψ is linear function to be optimized. The minimization is to be carried out, subject to constraints that the eigenvalues are not lower than $\zeta_0 > 0$ and the spring constants are nonnegative. Problem constraints are given in inequality constraint form as

$$\psi_1 = \zeta_0 - \zeta_1 \le 0, \qquad \psi_3 = -b_1 \le 0,$$
$$\psi_2 = \zeta_0 - \zeta_2 \le 0, \qquad \psi_4 = -b_2 \le 0, \tag{20}$$

Since the eigenvalues of Eq. (16) are $\zeta_1 = \left(4b_1 + 2b_2\right)/3$ and $\zeta_2 = 4b_1$, constraints (18) become:

$$\psi_1 = \zeta_0 - \frac{4b_1+2b_2}{3} \le 0, \qquad \psi_3 = -b_1 \le 0,$$
$$\psi_2 = \zeta_0 - 4b_1 \le 0, \qquad \psi_4 = -b_2 \le 0. \tag{21}$$

Equations (19) define a linear programming problem. The feasible set is shown graphically in Fig. 5. The slope of the line connecting points A and B is -2. The level lines of the cost function in Eq. (17) are straight, with slope equal to $-c_1/c_2$. The objective function decreases as level lines of objective move to the lower left. Thus, it is clear that point A (repeated eigenvalue) is optimum if $c_1 / c_2 > 2$ and point B (simple eigenvalue) is optimum if $c_1 / c_2 < 2$.

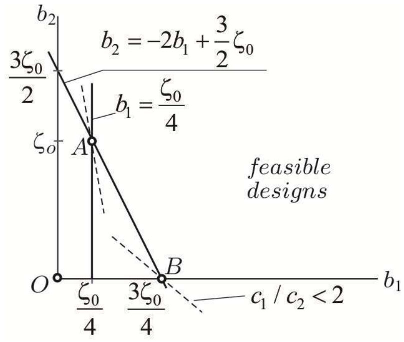

Fig. 5 Feasible region in design space for systems with two degrees of freedom

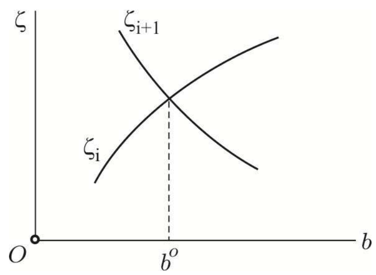

Fig. 6 Schematic of eigenvalue crossing

6. CONCLUDING REMARKS

The previous example is a simple demonstration how each eigenvalue would respond under a change. This may be physically explained by reference to Fig. 6, which shows how two eigenvalues ζ_i and ζ_{i+1} depend upon a system parameter b. For the structural dynamic analysis, further research is required to enhance the applicability of the nonlinear sensitivity analysis technique for high-frequency modes because, in some cases, this newly-developed technique may not converge as the predictions go beyond the limited bounds of Rayleigh quotient, and to improve the accuracy of this technique. The development of a structural optimization procedure which is capable of solving multiple natural frequency constraint problems is essential.

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